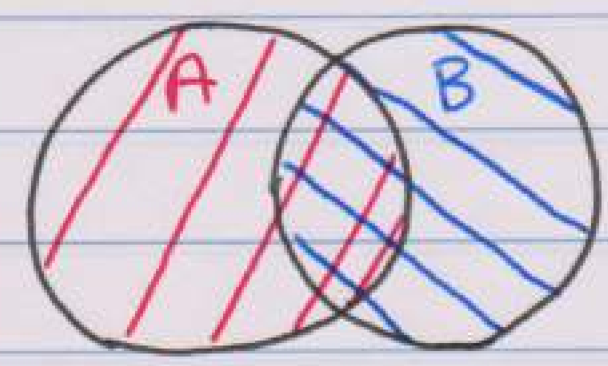
MATC44 Week 10 Notes

1. Inclusion - Exclusion Principle:

- Let $|A|$ denote the cardinality of set A. The inclusion- exclusion principle states that $|A \cup B| = |A| + |B| - |A \cap B|$. To understand this, consider the diagram below:



$|A \cup B|$ is the number of elements in set A and set B. But, doing $|A \cup B| = |A| + |B|$ will double count the elements in both set A and set B. Hence, we need to subtract $|A \cap B|$.

E.g. Let $A = \{1,2,3\}$
Let $B = \{3,4,5\}$
$A \cup B = \{1,2,3,4,5\}$
Note how 3 appears in both A and B but just once in $A \cup B$. Hence, if we don't subtract $|A \cap B|$, we will double count it.

- Similarly, $|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C|$.

– E.g. 1. How many natural numbers between 1 and 600 are there which are multiples of 2 or 3?

Soln:
There are 600/2 or 300 multiples of 2.
There are 600/3 or 200 multiples of 3.
There are 600/6 or 100 multiples of 6.
Hence, by I-E, there are 300+200−100 or 400 numbers that are multiples of 2 or 3.

Note: This is related to Euler's Function, denoted as ϕ. For any natural num, $\phi(n)$ is equal to the number of natural numbers $m \leq n$ which have no prime divisor with n. We can produce a formula for $\phi(n)$ using I-E.

– E.g. 2. Compute $\phi(n)$ if we assume that $n = (p_1)^{k_1} \cdot (p_2)^{k_2}$.

Soln:
Since p_1, p_2 are prime divisors of n, then there are $\frac{n}{p_1}$ multiples of p_1 and $\frac{n}{p_2}$ multiples of p_2 and $\frac{n}{p_1 \cdot p_2}$ multiples of p_1 and p_2.

$$\phi(n) = n - \left(\frac{n}{p_1} + \frac{n}{p_2} - \frac{n}{p_1 p_2}\right)$$

$$\phi(n) = n - \frac{n}{p_1} - \frac{n}{p_2} + \frac{n}{p_1 \cdot p_2}$$

$$= n\left(1 - \frac{1}{p_1} - \frac{1}{p_2} + \frac{1}{p_1 p_2}\right)$$

$$= n\left(1 - \frac{1}{p_1}\right)\left(1 - \frac{1}{p_2}\right)$$

– E.g. 3. Compute the number of all possible rearrangements of AAABBBCCC s.t. no 3 identical consecutive letters occur.

Soln:

We will subtract all the invalid rearrangements from the total number of rearrangements to get the answer.

Total number of rearrangements is equal to $\frac{9!}{3!\,3!\,3!}$ or 1680.

Next, consider the number of rearrangments s.t. there will always be 3 consecutive A's. It is equal to $\binom{7}{1} \cdot \frac{6!}{3! \cdot 3!}$

This is because there are 7 positions to place the first A and you're choosing one of them. There are now 6 spots left for the 3 B's and 3 C's. Hence, $\binom{7}{1} \cdot \frac{6!}{3! \cdot 3!}$

I.e. Consider the diagram below

$\underline{\quad}_1 \ \underline{\quad}_2 \ \underline{\quad}_3 \ \underline{\quad}_4 \ \underline{\quad}_5 \ \underline{\quad}_6 \ \underline{\quad}_7 \ \underline{\quad}_8 \ \underline{\quad}_9$

We can put the first A in any spot from 1 to 7. We can't put the first A in spot 8 or 9 because there's not enough room. So, $\binom{7}{1}$.

Note: The same logic and answer applies for re-arrangements s.t. there will always be 3 consecutive B's or C's. Hence, there are $3 \cdot \binom{7}{1}\left(\frac{6!}{3!\,3!}\right)$ or 420 ways we can make rearrangements s.t. there are 3 consecutive A's or B's or C's.

Now, consider the number of rearrangements s.t. we get 2 triple consecutive identical letters.
I.e. AAACCBBBC or AAACBBBCC
Let's find the number for 3 A's and 3 B's. To get the total number, we'll multiply it by 3 since we can have 3 A's and 3 B's or 3 A's and 3 C's or 3 B's and 3 C's.

For this, I'll denote AAA as A and BBB as B and there will be 5 spots instead of 9. The answer is $\binom{5}{1} \cdot \binom{4}{1}$ or 20. There's 5 ways to place A and 4 ways to place B. In total, there are 60 rearrangements where we get 2 triple consecutive identical letters.

Lastly, consider the case where we have 3 triple consecutive identical letters. There's 3! or 6 rearrangements to get this. To show this, I'll denote A as AAA, B as BBB and C as CCC. Furthermore, there are 3 spots instead of 9. We have 3 places for the first letter, 2 places for the second letter and 1 place for the third letter. In total, we have 3! or 6 permutations.

Therefore, the number of valid permutations is $1680 - 420 + 60 - 6$ or 1314.

2. Recurrence Relations:

- A very powerful method in combinatorics. It is used to compute all the states of a system that depends on n vars.

- Let a_n denote the num of all diff states that a system with n vars can take. Instead of computing a_n directly, we assume that a_{n-1}, and maybe a_{n-2}, are known and we compute a_n based on a_{n-1} (and a_{n-2} if needed)

- This technique is the analogue of the inductive principle. Hence, we want to obtain a relation of the following form:

a) $a_n = f(a_{n-1})$ OR

b) $a_n = f(a_{n-1}, a_{n-2})$

Solving a) requires knowing a_1 and solving b) requires knowing a_1, a_2. We will be using arithmetic progressions and geometric progressions to help solve recurrence equations.

- Arithmetic Progression:
  - Formula: $a_n = a_{n-1} + d$, where d is a constant.
  - Consider the following:

$$\left.\begin{aligned} a_2 - a_1 &= d \\ a_3 - a_2 &= d \\ &\vdots \\ a_{n-1} - a_{n-2} &= d \\ a_n - a_{n-1} &= d \end{aligned}\right\} n-1 \text{ equations}$$

By adding the $n-1$ equations together, we get $a_n - a_1 = (n-1)\cdot d$ or $a_n = a_1 + (n-1)\cdot d$.

Note: The above $n-1$ eqns are telescoping sums.

- Geometric Progression:
  - $a_n = r \cdot a_{n-1}$, where r is a constant.
  - Consider the following:

$$\underbrace{\frac{a_2}{a_1} = r, \frac{a_3}{a_2} = r, \ldots \frac{a_{n-1}}{a_{n-2}} = r, \frac{a_n}{a_{n-1}} = r}_{n-1 \text{ equations}}$$

These are telescoping products.

By multiply the $n-1$ equations, we get $\frac{a_n}{a_1} = r^{n-1}$ or $a_n = a_1 \cdot r^{n-1}$.

- E.g. 1. Compute the number of all subsets of the set $T_n = \{1, 2, 3, \ldots, n\}$.

Soln:

Let a_n be the number of all subsets of T_n. Now, consider $T_1 = \{1\}$ and a_1. $a_1 = 2$ because there are 2 subsets of T_1, the empty set $\{\}$ and the subset $\{1\}$. Similarly, for $T_2 = \{1,2\}$, $a_2 = 4$ because there are these 4 subsets: $\{\}, \{1\}, \{2\}, \{1,2\}$. We will show that $a_n = 2^n$ using the method of recurrent relations.

All subsets in T_n can be split into 2 categories:

a) All subsets in category a contains the element n.

b) All subsets in category b does not contain the element n.

Hence, any subset in category b is also a subset of T_{n-1}. This is because both sets contain all elements from 1 to $n-1$. Therefore, there are a_{n-1} subsets in category b. Furthermore, any subset in category a is a union of $\{n\}$ and a subset in category b. Hence, by the bijection principle, there are a_{n-1} subsets in category a.

Remark:

Consider $T_2 = \{1,2\}$. We will split all subsets into the 2 categories a and b.

Category b (Does not include 2):

$\{\ \}$, $\{1\}$ are the 2 subsets of T_n that belong in category b. They are the same subsets that appear for T_1.

Category a (Includes 2):

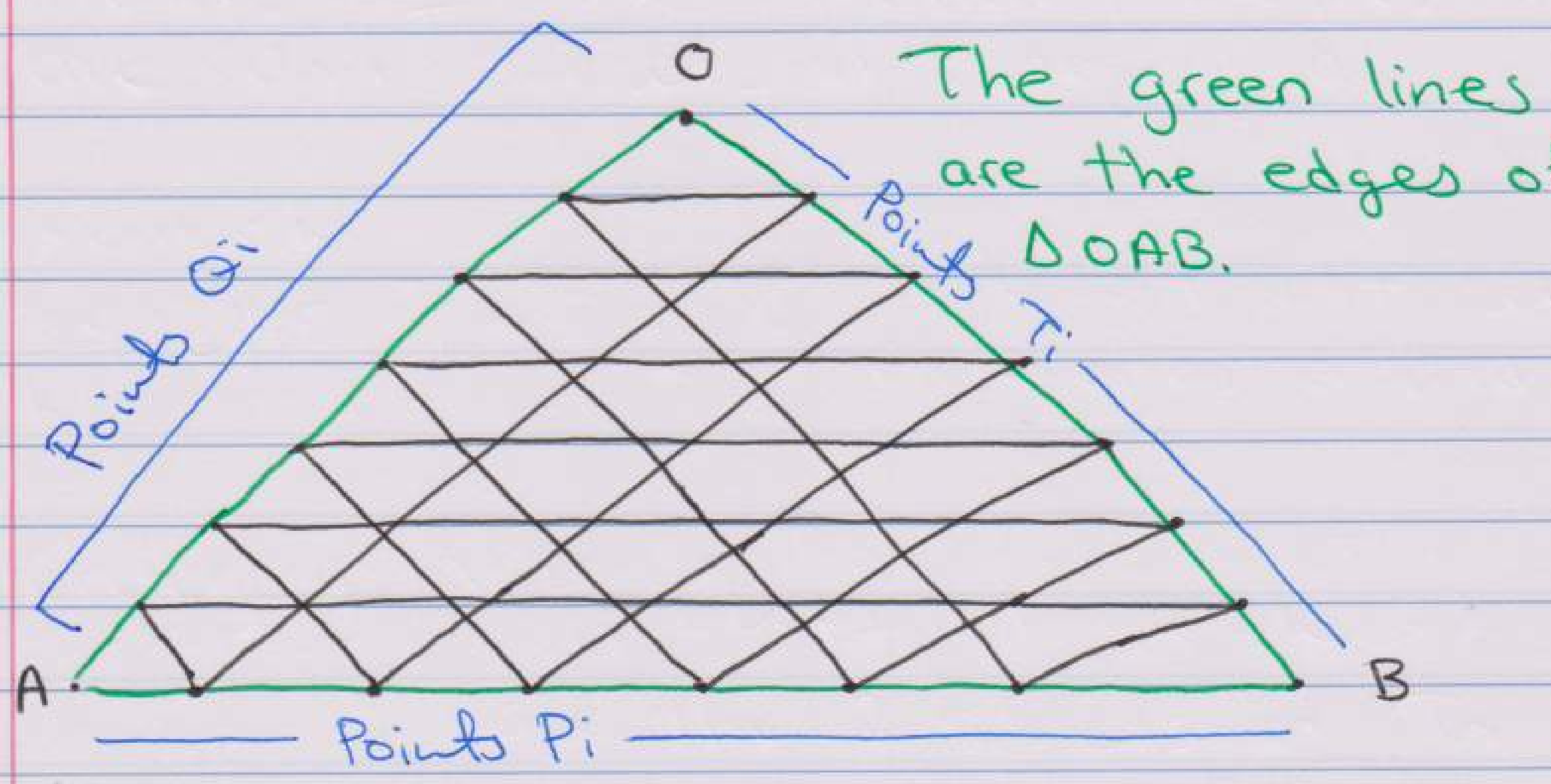
$\{\ \} \cup \{2\} = \{2\}$
$\{1\} \cup \{2\} = \{1,2\}$

Notice how there is a bijection between the 2 categories.

End of Remark

By the additive principle, $a_n = 2a_{n-1}$.
Since $a_1 = 2$, $a_n = 2^n$.

- E.g. 2. Consider an equilateral triangle OAB s.t. we add 6 points to each of the 3 edges and then we connect the points as such:



How many paths are there from O to A if:

a) We can only move left, right, left-down, and right-down. (We can't move up.)

b) We cannot visit a vertex more than once.

Soln:
Let a_n be the number we want to compute. Notice that once a path has reached a point on AB, then there is only 1 way to continue so that it terminates at A.

This implies that the number of paths from O to A is the same as the number of paths from O to any point on AB.

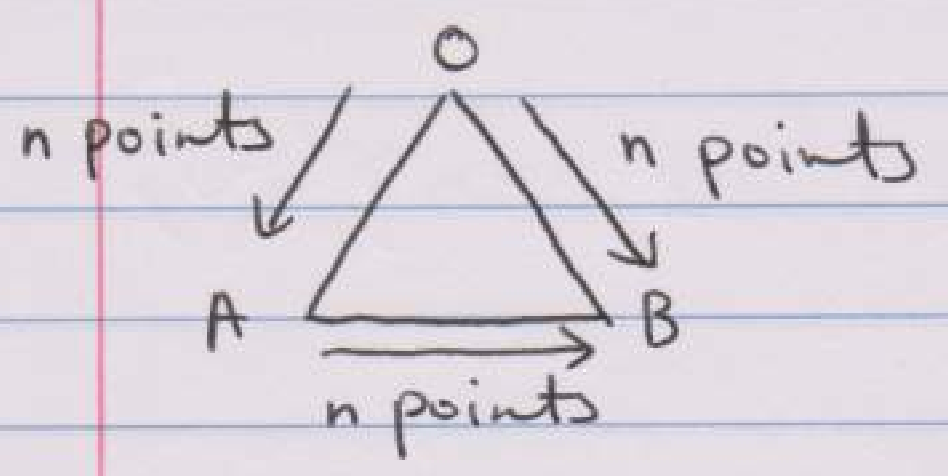
Remark:

Let $P_1, P_2, \dots, P_6$ be the 6 points on AB.
Let $O(P_i)$ denote the path from O to P_i, $i \in \{1,2,3,4,5,6\}$, s.t. the path terminates when it reaches P_i.
Let $E(P_i)$ denote the path from O to P_i, $i \in \{1, \dots, 6\}$, s.t. the path can still continue after reaching P_i.
For any P on AB, we get the following:
$|E(P)| = |O(A)| + |O(P_1)| + |O(P_2)| + \dots + |O(P_6)| + |O(B)|$

This is because $A, P_1, \dots, P_6, B$ are on the bottom row, and as such, there's only 1 way.

End of remark

Now, consider ΔOAB s.t. there are n points on each side.



That means there are $n+2$ points on the line AB. There are n points on the side plus the 2 points A and B. Hence, there must be $n+1$ points on the line immediately above AB. From this, we see the following:

a) $a_0 = 2$

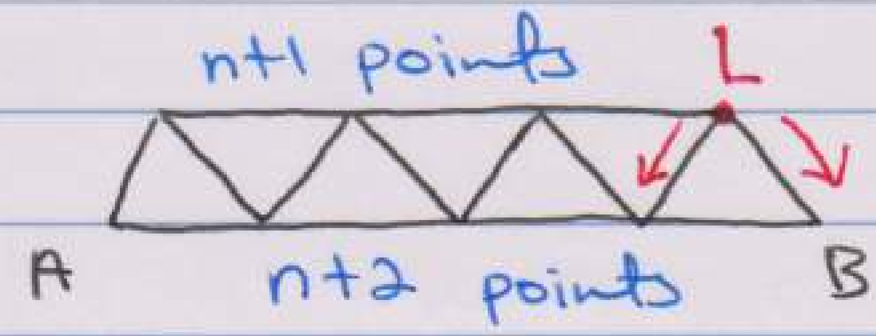
b) $a_n = (n+1)(a_{n-1})(2)$

For b, consider the following:

- There are a_{n-1} ways to reach that line and there are $n+1$ points to land on.
- We know that any path that reaches A must cross the second last line.
- From the second-last line, there are 2 directions we can go to reach A, down-left or down-right.

Remark:

Consider the diagram below.



There are a_{n-1} ways/paths to reach the 2nd last line, and there are $n+1$ points the path can land on. Then, there are 2 ways the path can land on line AB, down-left or down-right.

Suppose that the path landed at point L. From there, it can either go down-left or down-right to get to line AB. Then, the path would move left to reach A.

End of Remark

Since we know $a_0 = 2$ and $a_n = (n+1)(a_{n-1})(2)$, we can compute a_6.

$a_1 = 2 \cdot 2 \cdot 2 = 8$
$a_2 = 2 \cdot 8 \cdot 3 = 48$
$a_3 = 2 \cdot 48 \cdot 4 = 384$
$a_4 = 2 \cdot 384 \cdot 5 = 3840$
$a_5 = 2 \cdot 3840 \cdot 6 = 46,080$
$a_6 = 2 \cdot 46,080 \cdot 7 = 645,120$

– E.g. 3. Consider the triangle in eg 2, on page 9. How many parallelograms are formed by all segments that connect the points O, A, B, P_i, Q_i, T_i?

Soln:
Let the intersections of the segments be denoted as nodes. There are $1+2+\ldots+(n+1)+(n+2)$ or $\frac{(n+2)(n+3)}{2}$ nodes.

Let a_n be the num we want to compute.

All parallelograms can be divided into 2 categories:

a) Consists of all parallelograms that do not have a vertex on AB.

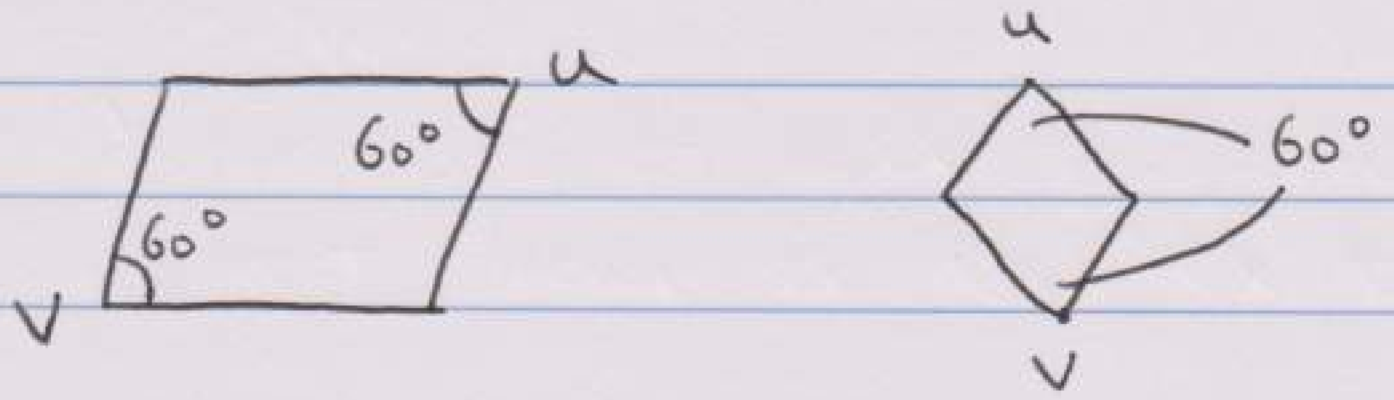
b) Consists of all parallelograms that have at least one vertex on AB.

Clearly there are a_{n-1} parallelograms of type a.

For type b, we will establish a bijection between all parallelograms of type b and specific pairs (u,v) of vertices s.t. v is one of the $n+2$ nodes on line AB. The bijection is the following: Any parallelogram has 2 opposite angles equal to 60 degrees and 2 opposite angles equal to 120 degrees. We will associate any parallelogram to the pair of vertices of the angles which are equal to 60 degrees. We will call any pair of vertices that come from a parallelogram in the above way admissible.

If (u,v) is admissible, then u lies in one of the 3 planar sections that are created by the 3 lines that pass through v and each of which has an angle of 60°. This implies that cannot lie on any lines that pass through v.

Remark:



End of Remark

We will next compute all admissible pairs (u,v) s.t. $v \in AB$. We have $n+2$ nodes on AB. For each of these $n+2$ nodes, call it v, we need to compute the total number of admissible nodes. We know one of the lines that pass through v is the line AB, which contains $n+2$ nodes. The number of nodes on the other 2 lines passing through v and excluding v is equal to $n+1$. Therefore, we have:

$$\frac{1}{2}(n+2)(n+3) - (n+2) - (n+1) = \frac{n(n+1)}{2}$$

admissible pairs for v. There are $n+2$ nodes on AB, and hence in total we have $\frac{n(n+1)(n+2)}{2}$ or $3\binom{n+2}{3}$ parallelograms of type b.

From above, we know that $a_n = a_{n-1} + 3\binom{n+2}{3}$. Furthermore, $a_1 = 3$, and $a_0 = 0$.

$$a_n = a_{n-1} + 3\binom{n+2}{3}$$
$$= a_{n-1} + 3\left(\binom{4}{3} + \binom{5}{3} + \dots + \binom{n+2}{3}\right)$$

By telescoping sums, we obtain:

$$a_n - a_0 = 3\sum_{k=0}^{n}\binom{k+2}{3} \quad \text{and hence}$$

$$a_n = 3\binom{n+3}{4}$$

Remark:

$$a_0 = 0$$
$$a_1 = a_0 + 3\binom{3}{3}$$
$$= 0 + 1$$
$$= 1$$
$$a_2 = a_1 + 3\left(\binom{4}{3}\right)$$
$$= a_0 + 3\binom{3}{3} + 3\binom{4}{3}$$
$$= 0 + 3\binom{3}{3} + 3\binom{4}{3}$$
$$= 3\left[\binom{3}{3} + \binom{4}{3}\right]$$
$$a_3 = a_2 + 3\binom{5}{3}$$
$$= 3\left[\binom{3}{3} + \binom{4}{3} + \binom{5}{3}\right]$$
$$\vdots$$
$$a_n = 3\left[\binom{3}{3} + \binom{4}{3} + \binom{5}{3} + \dots + \binom{n+2}{3}\right]$$

Note: $\binom{3}{3} + \binom{4}{3} + \dots + \binom{n+2}{3} = \binom{n+3}{4}$

I will prove this using induction.

Base Case:

$n=1$

$LS = \binom{3}{3} = 1$

$RS = \binom{1+3}{4}$

$= \binom{4}{4}$

$= 1$

$LS = RS$

Inductive Hypothesis:

Assume for all k, where $k < n+2$, that the equation holds.

Induction Step:

By IH, we know that $\binom{3}{3} + \binom{4}{3} + \ldots + \binom{n+1}{3} = \binom{n+2}{4}$.

$$\binom{n+2}{4} + \binom{n+2}{3} = \binom{n+3}{4}$$ by Pascal's Identity

$\therefore \ a_n = 3\binom{n+3}{4}$

End of Remark

– E.g. 4. How many diff sets of n pairs can be formed from 2n people.

Soln:

Let a_n be the number we want to compute.

Consider any person. There are $2n-1$ pairs that can be formed with this 1 person. After forming any of these pairs, we need to form $n-1$ pairs from the remaining $2n-2$ people. This can be done in a_{n-1} ways.
Hence, we get: $a_n = (2n-1)(a_{n-1})$
Furthermore, we know $a_1 = 1$.
From above, we get:

a) $a_1 = 1$

b) $a_2 = (2(2)-1) \cdot a_1$
$\quad = 3 \cdot a_1$

c) $a_3 = (2(3)-1) \cdot a_2$
$\quad = 5 \cdot a_2$

$\vdots$

d) $a_n = (2n-1) \cdot a_{n-1}$

By the method of telescoping products, we obtain that a_n is equal to the product of all odd numbers less than $2n$.

$$\frac{a_n}{a_{n-1}} = 2n-1$$

$$\frac{a_{n-1}}{a_{n-2}} = 2n-3$$

$$\vdots$$

$$\frac{a_3}{a_2} = 5$$

$$\frac{a_2}{a_1} = 3$$

From above, we obtain:

$$\frac{a_n}{\cancel{a_{n-1}}} \cdot \frac{\cancel{a_{n-1}}}{\cancel{a_{n-2}}} \cdot \ldots \cdot \frac{\cancel{a_2}}{a_1} = (2n-1)(2n-3)\cdots(3)$$

OR

$$\frac{a_n}{a_1} = (2n-1)(2n-3)\cdots(3)$$

And since $a_1 = 1$, $a_n = (2n-1)(2n-3)\ldots(3)$

Now, a_n can be rewritten as:

$$\begin{aligned} a_n &= (2n-1)(2n-3)\ldots(3) \\ &= \frac{(2n)(2n-1)(2n-2)(2n-3)\ldots(3)(2)(1)}{(2n)(2n-2)\ldots(2)} \\ &= \frac{(2n)!}{2(n)\cdot 2(n-1)\cdot \ldots 2(1)} \\ &= \frac{(2n)!}{2^n \cdot n!} \end{aligned}$$

– E.g. 5. We are given 3 pegs A, B, C and n disks of graduated size with holes in their center. Initially the discs are at peg A s.t. any disc is always on top of a bigger disk. We want to transfer all discs to another peg by moving 1 disk at a time and without placing a larger disk on top of a smaller disk. What is the min num of moves required to transfer the n disks?

Soln:

Let a_n be the num we want to compute. It takes a_{n-1} moves to transfer the first $(n-1)$ discs from peg A to peg B. Then, it takes 1 move to transfer the biggest disk from peg A to peg C. And it takes again a_{n-1} moves to transfer the $n-1$ disks from peg B to peg C.

Hence, $a_n = 2a_{n-1}+1$
$a_1 = 1$

To solve this recurrence relation, I will add 1 to both sides and obtain
$a_n+1 = 2(a_{n-1}+1)$.

If I denote $x_n = a_n+1$, then $x_n = 2x_{n-1}$, with $x_1 = a_1+1 = 2$. Hence $x_n = 2^n$ and $a_n = 2^n-1$.

— E.g. 6. How many subsets of the $T_{2018} = \{1, 2, \ldots, 2018\}$ are there s.t. the sum of the elements in the subset is even?

Soln:

Consider $1+2+\ldots+2018 = \frac{2018 \cdot 2019}{2} = 2,037,171$

This number is odd. Now, consider any subset that has an even sum. We know that the sum of the remaining numbers must be odd. This is because even + odd = odd. Hence, there is a bijection between

the subsets with an even sum and the subsets with an odd sum.
Hence, the answer is 2^{2017} or $\frac{2^{2018}}{2}$.